18th IMO 1976

A1. A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.

A2. Let P1(x) = x2 - 2, and Pi+1 = P1(Pi(x)) for i = 1, 2, 3, ... Show that the roots of Pn(x) = x are real and distinct for all n.

A3. A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.

B1. Determine the largest number which is the product of positive integers with sum 1976.

B2. n is a positive integer and m = 2n. aij = 0, 1 or -1 for 1 = i = n, 1 = j = m. The m unknowns x1, x2, ... , xm satisfy the n equations:

ai1x1 + ai2x2 + ... + aimxm = 0,

for i = 1, 2, ... , n. Prove that the system has a solution in integers of absolute value at most m, not all zero.

B3. The sequence u0, u1, u2, ... is defined by: u0= 2, u1 = 5/2, un+1 = un(un-12 - 2) - u1 for n = 1, 2, ... Prove that [un] = 2(2n - (-1)n)/3, where [x] denotes the greatest integer less than or equal to x.