19th IMO 1977

A1. Construct equilateral triangles ABK, BCL, CDM, DAN on the inside of the square ABCD. Show that the midpoints of KL, LM, MN, NK and the midpoints of AK, BK, BL, CL, CM, DM, DN, AN form a regular dodecahedron.

A2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

A3. Given an integer $n > 2$, let Vn be the set of integers $1 + kn$ for k a positive integer. A number m in Vn is called indecomposable if it cannot be expressed as the product of two members of Vn. Prove that there is a number in Vn which can be expressed as the product of indecomposable members of Vn in more than one way (decompositions which differ solely in the order of factors are not regarded as different).

B1. Define $f(x) = 1 - a \cos x - b \sin x - A \cos 2x - B \sin 2x$, where a, b, A, B are real constants. Suppose that $f(x) = 0$ for all real x. Prove that a2 + b2 = 2 and A2 + B2 = 1.

B2. Let a and b be positive integers. When a2 + b2 is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs a, b such that q2 + r = 1977.

B3. The function f is defined on the set of positive integers and its values are positive integers. Given that $f(n+1) > f(f(n))$ for all n, prove that $f(n) = n$ for all n.